New Directions in Economic Justice

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Roger Skurski

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Justice and Mathematics: Two Simple Ideas

ROBERT D. COOTER

INTRODUCTION

Mathematics is the perfection of analytical skill just as poetry is the perfection of the imagination. In mathematics we pass from one proposition to another according to explicit rules, whereas in poetry we pass from one image to another without rules to guide us. Mathematical reasoning is constrained by well-defined concepts of consistency, whereas the play of imagination in poetry is constrained by sensibility or taste.

Is justice a sensibility or can it be analyzed mathematically? Political justice resides in the allocation of duties and rights to individuals by law and political practice. These allocations are complex, but the political theories advanced to explain them are simple. We shall show that two traditions of political theory are erected upon two simple mathematical ideas.

The idea of a contract was raised to the level of political theory by the contractarians or social-contract theorists, such as Hobbes, Locke, and Rousseau. Contractarians conceive of the state as a bargained agreement analogous to an ordinary business contract. From this perspective the state is viewed as a cooperative venture for mutual gain.

The idea of a maximum was raised to the level of political theory by the utilitarians, especially Bentham. The utilitarians conceive of the state as maximizing social welfare, which is analogous to a private firm maximizing profits. The contractarian and utilitarian political philosophies are the two traditions which we shall try to explain with the help of simple mathematics.

These two traditions do not claim any true believers among contemporary lawmakers. However, contractarianism and utilitarianism are part of the intellectual culture pervading the law. The classical contractarian texts are a reference point for interpreting the American constitution, and utilitarianism motivated many reforms of British law. Lawmakers reflect upon difficult decisions from the vantage point of these traditions. It is helpful to think of contractarianism and utilitarianism as deep metaphors for thinking about almost any legal issue.

Our aim is to clarify these two metaphors by describing their mathematical structure. It is characteristic of mathematics that it illuminates political disputes without resolving them. Although our analysis cannot resolve disputes about justice and the law, it can increase the quality of the debate by making the disputants lighter on their feet.

Before beginning, a comment is due on the relationship between describing the law and criticizing it. The positive-normative distinction is one line of demarcation between science and nonscience in positivist methodology. For example, some economists believe that economics can be separated from ethics by separating explanation from evaluation. This boundary receives little respect in the work of most utilitarians or contractarians. For example, Bentham passes from explanation to evaluation with ease because his aim is to rationalize the law, i.e., to criticize the parts that do not conform to the fundamental principles. In his view the law as a whole conforms to the principle of utility, although parts of it diverge. Thus the principle of utility can be used to explain the basic structure of the law and to criticize the laws that are inconsistent with its fundamental design. This essay concerns the attempt by utilitarians and contractarians to rationalize the law.

The essay is divided into two parts. First, the simple mathematical ideas underlying utilitarianism and contractarianism are explained, and the historical antagonism between these philosophies is explained by the tension between the mathematical ideas (Part I). The second step is to relate these two philosophies to the fundamental structure of the law, which is accomplished by connecting the
A useful artifice of economic theory is to imagine a timeless and certain world. We replace the complexity of real choices with "time slices" that have no past or future. These time slices are called "states of the world," which consist of detailed descriptions of nature and society. There are many possible states of the world $s_1, s_2, s_3$, etc., where the $s$ indicates a state of the world and the subscript indicates which one it is. The basic idea in economic theories of choice is that states of the world can be ranked according to their overall goodness or desirability. For example, the set of states listed in Figure 1 might be ranked $s_1 < s_2 < s_3$, where the states get better as we move from left to right.

Many relations between objects are orders; for example, "heavier than" or "faster than." There are also many unordered relations; for example, squash players find that player A can beat B, and B can beat C, but C can beat A. Each of the three players beats someone and loses to someone, so we cannot set them in order of skill and identify a best player. A relation that cannot be ordered is called "intransitive." Microeconomic theory assumes that overall goodness is like weight and unlike skill at squash: A rational person can set states of the world in order of goodness.

As soon as we accept transitivity as an axiom of rationality, so that states can be ordered, there is a drift to accept maximization as the basis for choice. After all, why do worse when you can do better? It would be irrational to choose a state that is worse when a better one is available.

The first principle of decision theory in economics is to maximize goodness subject to the constraints inherent in the choice situation. For example, firms judge goodness by profits alone in the conventional formulation, so firms maximize profits subject to the constraints imposed by the technology of production. A constrained
maximum can be represented by a saddle point in a two-dimensional diagram (Figure 2), which is familiar to any student of introductory economic theory. There is no habit of thought among microeconomists which is more compelling than the urge to characterize each human choice as a saddle point on an appropriately chosen pair of axes.

Transitivity connects a theory of value to the real number system, and maximization connects it to calculus and related optimization techniques. It is possible in principle to have a theory of a value which is ordered, in the sense of being transitive, but which does not postulate maximization. To my knowledge, philosophers have not advanced such theories. Economists and psychologists have made the attempt through so-called satisficing or learning-behavior models, but these beginnings do not compare in scope or power to the utilitarian tradition. The utility theory of choice, as incorporated in economics, is the outstanding form of value theory based upon the real numbers.

We have explained the mathematical idea of maximization which is fundamental to utilitarianism. Now we turn to contractarianism. The theory of contract in jurisprudence has focused upon the questions, “When does a contract exist?” and “What are its terms?” For example, the proposal has been made that a contract comes into existence when each side offers the other a tangible benefit (called “consideration”) or the promise of it. The terms of a contract are taken to be those explicitly stated in writing or words, plus those that can be imputed by virtue of established conventions and practices.

The criteria for existence and identification of contracts, which are central to jurisprudence, are taken for granted in economics. Economists usually assume that a wide range of possible contracts are available, each with well-defined terms. The relevant problem of contract which economists have addressed could be called “contract selection,” i.e., “What will be the terms on which individuals will strike a bargain?” Economists try to predict the terms of the contract which will prevail at the end of the bargaining process.

Economists have developed their own jargon for describing bargaining. The level of welfare that a person can achieve on his own without entering into a contract with the parties in question is called his “threat value.” Each person must receive at least his threat value, or there would be no advantage for him in entering the agreement. A bargaining situation is viable if both parties could benefit from cooperation. The problem of contract selection studied by economists is to predict the likelihood and terms of cooperation. There is a question of efficiency—namely, “Will the parties cooperate?”—and a question of distribution—namely, “How will the advantages of cooperation be divided among individuals?”

Some economic theories assume that bargaining can be described in money terms. For example, consider a business contract between, say, a cotton grower and a textile manufacturer. The threat value is the profit level that each can achieve without exchange between them, e.g., the manufacturer’s profit when buying cotton from the best alternative source. The sum of the grower’s and manufacturer’s threat values is the “noncooperative value” of the game. The “cooperative value” of the game is the sum of profits under a contract whose terms maximize the joint profits of the grower and the manufacturer. The surplus from cooperation is the difference between the cooperative and the noncooperative values of the game. The bargaining problem is to find terms for splitting the surplus which both parties will accept.

The solution to this bargaining problem is called a “bargaining equilibrium.” The exact nature of this equilibrium is a subject of dispute, but we can explain the equilibrium concept in abstract terms. A game situation is an equilibrium if no player wishes to revise his move. Let \((x_1, x_2, \ldots, x_n)\) describe the moves of players 1, 2, \ldots, \(n\), respectively. Let \(f\) describe how each player revises his move in light of what other players are doing. \(f\) is called a “reaction function” because it describes how players react to each other. For example, the moves \((x_1, x_2, \ldots, x_n)\) may cause the reaction \((x_1', x_2', \ldots, x_n')\) as specified by \(f\):

\[
(x_1', x_2', \ldots, x_n') = f(x_1, x_2, \ldots, x_n).
\]

An equilibrium is a situation where no player wishes to revise his move given what the other players are doing, i.e., and \((x_1^*, x_2^*, \ldots, x_n^*)\) such that

\[
(x_1^*, x_2^*, \ldots, x_n^*) = f(x_1^*, x_2^*, \ldots, x_n^*).
\]

From a mathematical standpoint, an equilibrium is a fixed point in the reaction function \(f\).

The most familiar example of a fixed point in economics is the equilibrium of a competitive market, which is represented by the intersection of supply and demand curves. This example is not useful for our purposes because there is no scope for bargaining in competitive markets. But we can use a less familiar economic model, the Cournot duopoly game, to illustrate a bargaining equilibrium.
Consider Figure 3. Suppose Alphonse and Gascon operate gas stations across the street from each other. Imagine that they are bargaining with each other in an attempt to set prices cooperatively. One curve in Figure 3 depicts the price that Alphonse will favor in reaction to Gascon's suggested price, and the other curve represents the price that Gascon will favor in reaction to Alphonse's suggested price. Starting from any arbitrary prices, the reactions of the players cause the prices to converge to the intersection of the lines, as illustrated by the sequences of prices \((P_0, P_1, P_2, P_3, P_4, P_5, \ldots)\). The point of intersection is the equilibrium or fixed point in the reaction functions.

A fixed point and a saddle point are mathematically distinct ideas. A fixed point presupposes a measure of value which is maximized. For example, in economics the behavior of a firm is described as a saddle point, i.e., a point where profits are maximized. However, the interaction of firms in duopoly is described as a fixed point, i.e., an equilibrium in the reaction of firms to each other.

It matters whether we think of politics as concerned with a maximum equilibrium or a fixed-point equilibrium. Under the first description we think of government as if it were disciplined to pursue a common purpose, like a business firm pursuing profits. Under the second description, we do not think of political life as having a single, shared purpose. Rather, we think of politics as an agreement specifying the extent to which different individuals are entitled to pursue their private ends. The utilitarian conception emphasizes harmony and common values, whereas the contractarian conception emphasizes restraint and private goals. This contrast is implicit in the difference between maximizing value and an equilibrium in the reaction function of individuals. We shall use this distinction in mathematical ideas to compare the utilitarian and contractarian traditions in political theory.

B. The Structure of Utilitarianism

The first systematic utilitarian was Jeremy Bentham, who was twenty-five years younger than Adam Smith. We begin by reviewing some salient features of Bentham's cardinal utility theory. First, Bentham identified the utility of a situation with a definite quantity of pleasure or pain produced by it. His concept of pleasure was broad, encompassing such similar (but different) ideas as benefit, happiness, advantage, and goodness. He discussed various qualities of pleasure which affect the quantity of it afforded by particular circumstances, such as intensity, duration, and certainty. The total pleasure afforded by a situation can be found by adding up the pleasure from its different aspects, after allowing for these qualities. No special problems arise if the pleasure belongs to more than one person. Bentham believed that the pleasure of different individuals can be added together just like their body weights can be summed. He gave no instructions for how to carry this out, which is an unfortunate tradition maintained by his intellectual heirs in philosophy.

A second feature of Bentham's theory is the equity rule, "Each to count for one and none to count for more than one." This rule requires everyone's pleasure to be given equal weight in social choice. Bentham did not pay much attention to this equity rule; in fact, the above statement of it is by Mill. In Bentham's view the idea that one person's pleasure could count more than another's is a misunderstanding of the metric, rather like the confusion that an ounce of lead weighs more than an ounce of feathers. Once we understand the metric, there is no further problem of fairness among individuals. In utilitarianism, efficiency and equity get collapsed into the idea of a single metric that measures social welfare. The concept of justice in Benthamism is the concept of a correct measure.

The equity rule introduces a bias toward income redistribution in utilitarian thinking. Bentham noted that the pleasure from spend-
money depends upon the intensity of the wants satisfied by the expenditure, and poor people have more urgent unsatisfied wants than rich people. The implications of such an outlook for economics were fully developed by A.C. Pigou. According to Pigou, there is no precise way to compare one person’s pleasure with another’s, but all the evidence of common sense directs economists to proceed on the assumption that a dollar is more valuable to a poor person than to a rich person. Pigou concluded that economists should recommend policies that distribute national income more equally without appreciably diminishing its sum total. Thus the requirement that each person’s pleasure receive equal weight was egalitarian in its implications.

A third feature of this doctrine is that it is strictly forward-looking, by which I mean that there is rigorous application of the maxim “Bygones are bygones.” Only the present and future can afford us pleasure; we have no access to retroactive enjoyment. In order to act we need to know how much present and future pleasure alternative actions will afford. That is all we need to know. The past is fully represented in our decision by the causal propensity of the present to afford pleasure. Past actions cannot create obligations or duties that are distinct from causal propensities. Philosophers call such a theory “consequentialist,” because decisions are guided by the action’s consequences, but not by its antecedents. This feature of utilitarian doctrine represents a reforming and critical spirit toward the past, which prompted many proposals to reform the law.

The final feature of Bentham’s doctrine is that actions can be ranked according to the magnitude of the pleasure which they afford. Since all situations can be ranked according to pleasure, and pleasure is the ultimate standard of goodness, it follows immediately that rational behavior requires maximizing pleasure. The individual is deemed as maximizing his own pleasure, and the interaction of individuals is described as maximizing the sum of individual pleasures. This optimistic account of the ecology of interaction, according to which self-interested individuals accomplish social goals, is reminiscent of Adam Smith’s account of how private greed serves public purposes in free markets.

One of the glaring faults in Bentham’s philosophy is his failure to justify this optimism. Adam Smith had described an “Invisible Hand” that directs self-interested traders to benefit the public by exchanging in competitive markets. This argument anticipated contemporary proofs that individuals who maximize their own utility by exchanging in a competitive market will reach an equilibrium which is efficient. Bentham possessed one of the fundamental concepts in this proof—namely, maximization—but not the other—namely, equilibrium. Consequently, he did not perceive clearly the need to identify a mechanism comparable to the hidden hand that operates in government. There is no account of a mechanism that guides self-interested individuals in politics or the courts to act so that social welfare will be maximized.

By thinking of pleasure as a quantity to be maximized, Bentham established a link between rational choice and the calculus. The exploitation of this linkage by Bentham’s successors established the pre-eminence of utilitarian ethics in economic reasoning.

In summary, Bentham’s utilitarianism has four attributes discussed in ethics: (1) hedonism (broadly conceived); (2) egalitarianism; (3) consequentialism; and (4) maximization. Attribute (4) permits government to be described as a saddle point, and attributes (1)-(3) characterize the metric. These four attributes have a natural affinity for each other, rather like the elements of an art style. For example, Greek columns go with mathematically symmetrical buildings, and income redistribution goes with maximizing social welfare. If we think of government as a saddle point, then we tend to think of the metric as possessing attributes (1)-(3), although it is possible to separate them.

C. The Structure of Contractarianism

The social-contract tradition is an aggregation of related theories without a dominating figure comparable to Bentham. There is room for disagreement about the most salient features of the tradition. We shall describe features of contractarian thought which correspond perfectly to the economic model of bargaining and imperfectly to any particular contractarian doctrine. In effect we consider an abstract social-contract doctrine that is purified of its non-economic elements, and we use this abstraction to classify the major contractarian philosophers. Our approach provides a unified perspective on a complex tradition, but we do not claim to uncover the hidden unity in diverse philosophies. In order to avoid false appearances, we shall refer to the doctrine which we explain as “economic contractarianism.”

A familiar definition of the state is the institution that possesses a monopoly of coercive force. Contract theories can be viewed as explaining why people would be willing to create such a monopoly.
There are three steps in developing this argument. The first step is to find out what people would do in the absence of civil government, which is aptly called the "state of nature." The second step is to characterize the advantages available to them by creating a state. The third step is to show the terms for distributing the advantages of government. The terms of the contract are described concretely in the fundamental laws or constitution of the state.9

The three steps in constructing this political contract correspond to elements in the economic theory of business contracts. The state of nature is the noncooperative solution that prevails if the parties cannot agree. Civil society is the cooperative solution. The social surplus is the difference between the level of welfare in civil society and the state of nature. The political contract specifies the terms for dividing the surplus from cooperation. Cooperation can improve the well-being of everyone in the sense that each person can be raised above the standard of life that he enjoys in the state of nature. We use the phrase "economic contractarianism" to refer to any contract theory containing these three steps.

According to game theory, a rational player will bargain with the aim of achieving agreement on terms favorable to himself. The level of well-being that a person can achieve on his own, without the cooperation of others, is his threat value. In social-contract theory the threat value of an individual or group is what they could secure without the protection of government. The stronger the threat, the more the person or group must be given to make cooperation worthwhile. Bargaining theory predicts that individuals and groups will enjoy legal rights and advantages in proportion to what they could secure for themselves without the help of government.

We identified four features of Bentham's utilitarianism: (1) Pleasure measures value (hedonism); (2) each person's pleasure receives equal weight (egalitarianism); (3) the past is represented by the causal propensities of the present (consequentialism); and (4) value should be maximized (maximization). Let us contrast these attributes to a social-contract theory modeled upon the economic theory of bargaining.

Utilitarianism asserts that social arrangements tend to maximize the sum of individual pleasures. If the pleasures of different individuals can be summed, then they must be commensurable, i.e., the progress of individuals toward different goals must be reducible to a single measure of value, namely, utility or pleasure. By contrast, economic contractarianism postulates a reaction function describing how people bargain. Each person bargains so as to maximize progress toward his own goals, but the reaction function does not require private goals to be commensurable. One person may seek wealth, another may seek political power, and a third may seek scientific truth. There is no commitment to the view that disparate goals can be reduced to the same thing, namely, pleasure. In this respect contractarianism is amenable to ethical relativism and utilitarianism is not.

In most economic theories, bargaining achieves an equilibrium when the scope for mutual gain is exhausted. Equilibrium is typically a situation where no one can be made better off without making someone else worse off. Economists call this Pareto efficiency. A bargained agreement is likely to be Pareto efficient.20 Pareto efficiency is a concept of value that is intended to avoid the problem of incommensurability. A Pareto optimum is not necessarily a maximum on a shared standard of value. Though the political contract may be described as Pareto efficient, it does not necessarily maximize anything.

Social cooperation produces income and wealth that must be distributed to individuals. The utilitarian equity rule requires each person's pleasure to receive equal weight when distributing income and wealth. By contrast the economic theories of contract distribute the surplus from cooperation according to threat values in the state of nature. If threats are equal in nature, then rights are equal in society. The extent and form of civil inequality depend upon the extent and form of natural inequality. The political contract is egalitarian to the extent that the state of nature is egalitarian, but not more so.

We called utilitarianism "consequentialist" because the decision maker is supposed to be guided by the consequences of his actions, not by their antecedents. The consequences are completely described in their relevant aspect for utilitarians by the amount of pleasure produced. By contrast the antecedents of an action have a direct and compelling influence upon decision makers who follow a contractarian philosophy. The task of such government officials is to carry out the terms of the political contract, not to increase social welfare. Government officials are asked by contractarians to guide their behavior by looking back to the political contract, not forward to the sum of pleasures. The political contract may be described as the logical antecedent to the state. Thus the distinction is between guiding choice by its effects or guiding choice by its logical antecedents.

Obviously, no government official should be blind to future consequences or past commitments. A well-articulated political philosophy must contain forward-looking and backward-looking
The claim that utilitarians guide action by its consequences, and contractarians guide action by its antecedents, concerns the most fundamental orientation of government officials. The dispute concerns whether the fundamental guide for state action is future social welfare or a past social contract.

We have discussed the attributes of a social-contract theory that is purified by omitting features that do not belong to the economic theory of games. There are four attributes of such a theory which are antithetical to the four attributes of utilitarianism:

1. Value is private and relative; there is not necessarily a shared value such as pleasure.
2. Civil equality exists only to the extent of natural equality.
3. Collective choice is ultimately guided by its logical antecedents, not by its causal consequences or effects upon welfare.
4. The political contract is Pareto efficient, but a Pareto optimum is not necessarily a maximum on a shared standard of value.

The difference in attributes can be explained by the difference in the simple mathematical ideas underlying the two philosophies. In utilitarianism the individual and society are described by the same concept: maximization. Maximization presupposes a measure of value. Consequently, utilitarianism assumes that state action can be measured against a shared standard of value. The state is regarded as a kind of superperson who resolves conflicts of interest among individuals in the same way that an individual supposedly resolves conflicts among his own motives, i.e., by maximizing pleasure. In contractarianism the individual and society are described by a different concept: The individual may be a maximizer, but the state is not. The state is regarded by economic contractarians as a kind of political market in which rights are exchanged. Contractarianism allows for more tension among individuals by allowing their goals to be incommensurable, and the incommensurability of individual goals prompts skepticism about collective action.

**D. A Parable in Arithmetic**

We have shown that the distinction in mathematical ideas underlying utilitarianism and economic contractarianism explains the contrast in their attributes. It is useful to illustrate this claim in a concrete example. Imagine a two-person world in which there are three activities: farming, defending, and robbing. The amount of corn that is produced depends upon how much time each person spends farming, and upon whether the mode of farming is cooperative or noncooperative. More is produced by cooperation because there is no need to devote time to defending or robbing. The exact numbers are given in Figure 4.

We wish to determine the consumption of each person in the state of nature and in civil society. From the figure we see that total production under noncooperation is 150, whereas by cooperating together the joint product is 250. The gain or surplus from cooperating is 100. The economic problem of forming a social contract is to distribute this surplus.

**Figure 4**

The Production and Distribution of Corn

<table>
<thead>
<tr>
<th></th>
<th>1st person</th>
<th>2nd person</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solitary Output</td>
<td>100</td>
<td>50</td>
<td>150</td>
</tr>
<tr>
<td>Non-cooperative</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gain from Theft</td>
<td>+10</td>
<td>+20</td>
<td>+30</td>
</tr>
<tr>
<td>Loss from Theft</td>
<td>-20</td>
<td>-10</td>
<td>-30</td>
</tr>
<tr>
<td>Solitary Consumption</td>
<td>90</td>
<td>60</td>
<td>150</td>
</tr>
<tr>
<td>Cooperative</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Joint Output</td>
<td>0</td>
<td>0</td>
<td>250</td>
</tr>
<tr>
<td>Hobbesian Consumption</td>
<td>140</td>
<td>110</td>
<td>250</td>
</tr>
<tr>
<td>Lockeian Consumption</td>
<td>150</td>
<td>100</td>
<td>250</td>
</tr>
<tr>
<td>Rawlsian Consumption</td>
<td>125</td>
<td>125</td>
<td>250</td>
</tr>
</tbody>
</table>

Economic theorists are not unanimous about how the surplus will be distributed, but the simplest solution concept is due to Nash. According to Nash, each player will receive his threat value plus an equal share of the surplus. An equal share of the surplus allocates 50 to each player. Suppose that the parties would steal from each other in the absence of government. From the figure we see that if both engage in theft, the first party consumes 90, and the second party consumes 60. The threat values when the parties resort to mutual plunder are 90 and 60. Thus we can compute the equilibrium consumption by adding 50 to the threat values, yielding the distribution 140 and 110, which is labeled "Hobbesian Consumption" in Figure 4. This distribution describes the economic allocation under a social contract derived under the assumption that the state of nature involves mutual plunder.
Another possibility is that individuals would not engage in un-
regulated plunder if government were absent. Perhaps the players
would recognize certain natural rights in property. Even animals
fight more vigorously when they defend their own territory. We
could imagine that theft would be censured in the state of nature on
moral grounds, even though no legal apparatus existed to enforce
morals. Suppose that the parties could threaten nonparticipation in
joint production, but not theft, when negotiating the social contract.
We see from Figure 4 that solitary output is 100 and 50, which rep-
resents the consumption levels the parties can achieve on their own
in the absence of theft. Thus the threat values when the parties resort
to nonparticipation, but not theft, are 100 and 50. We can compute
consumption under a bargained equilibrium with nonparticipation
as the threat by adding 50 to the threat values, yielding the distribu-
tion 150 and 100, which is labeled “Lockean Consumption” in Fig-
ure 4. This distribution describes the economic allocation under a
social contract derived under the assumption that the state of nature
involves nonparticipation (solitary production, no theft).

In anarchy individuals would benefit from natural strengths
such as intelligence and a strong body. Unequal outcomes in anarchy
imply unequal threats, and unequal threats result in inequality in
the bargained equilibrium. The two kinds of social contract dis-
cussed so far result in unequal economic distribution. Suppose that
someone wished to build a contractarian theory that did not allow
individuals with superior natural strength to bias the laws of distri-
bution in their favor. How would such a theory be constructed? By
changing the conception of the state of nature. Rather than thinking
of nature as anarchy, we think of nature as a condition of primitive
equality. We ask ourselves, “What would be the principles of distri-
bution if all unfair advantages that result from inequalities in natural
strength were removed?” If threats of plunder or nonparticipation
were not allowed, then the parties represented in Figure 4 would not
have any basis for claiming an unequal share of the cooperative
product. Consequently the total cooperative product would be split
equally among them, yielding the distribution 125 and 125, which is
labeled “Rawlsian Consumption” in Figure 4. This distribution
describes the economic allocation under a social contract derived
under the assumption that the state of nature involves primitive
equality.

The methods by which a utilitarian would solve the problem of
choosing a principle of distribution are different from those of the
economic contractarian. A utilitarian would begin by identifying

Bentham. The maximum occurs when the marginal utility from
consuming a small additional amount of corn is the same for both
individuals. If both people have similar needs (i.e., if they are “equally
efficient pleasure machines”), then they will both have the same
functional relationships between utility and corn. The sum of utili-
ties would be maximized under this condition when each consumes
an equal amount of corn, yielding the allocation 125 and 125.

The utilitarian solution to the parable in arithmetic (1) assumes
that each person’s consumption of corn can be reduced to the same
measure of social value (utility or pleasure); (2) gives equal weight
to each person’s pleasure; and (3) maximizes total pleasure. The eco-
E. Three Contractarian Philosophies

The names used to describe the three distributions in the parable in arithmetic, Hobbesian, Lockean, and Rawlsian, are chosen because the structure of the argument is reminiscent of these philosophers. In Hobbes's philosophy, natural behavior is not influenced by legal or moral norms, rapaciousness is unrestrained, it's every person for himself, there is a “war of all against all.” The definitive feature of this version is that dissolution of civil restraints results in dissolution of moral restraints.

In Locke's philosophy, people in a state of nature perceive themselves as possessing natural rights and duties, independent of the legal system. People fight each other in order to protect these rights or as retribution for their violation. Behavior is normative, not rapacious, because people recognize that others have rights similar to their own. The psychology is different from nature as normlessness, because the motive for harming others is to protect what is yours rather than to get as much for yourself as possible. Though some measure of peace and security is possible in Locke's state of nature because people practice moral restraint, the level of security is less than in civil society. The motive for erecting the state is to provide a more firm guarantee of each person's rights than anyone can assure by acting on his own or through private associations.

In our parable in arithmetic the Hobbesian distribution was reached by threatening mutual plunder, whereas the Lockean distribution was reached by threatening nonparticipation. The names were chosen because they are characteristic of Hobbes's philosophy that the state of nature involves normlessness, whereas the state of nature in Locke's conception involves moral norms but not statutes. The mathematical-bargaining theory used in our parable is contemporary, which is one reason why we have not claimed to offer a strict interpretation of these classical philosophies. Our use of bargaining theory elucidates these classical philosophies by clarifying one strand of thought found in them.

Hobbes and Locke described the state of nature as the historical condition of tribal people. These classical texts are cluttered with anthropological claptrap. However, the historical interpretation can be abandoned by giving the social contract a hypothetical interpretation. The dissolution of the civil state may be adopted as a goal at any time. Each generation reaffirms the social contract by rejecting anarchy and working to preserve or improve the state. The social contract is regarded as the logical antecedent to citizenship, not the historical antecedent to the state.

Plunder (Hobbes) and nonparticipation (Locke) do not exhaust the list of possible antecedents to citizenship. There is a long tradition in philosophy, reaching back to Plato's Republic, which holds that the first virtue of government is justice. Rather than thinking of the state of nature as a condition that would actually exist if government dissolved, we could think of the state of nature as a condition that would exist if all the unjust advantages that some people enjoy over others were removed. This is how Rawls approached the rehabilitation of contractarianism. Rawls describes an original bargaining position in which a veil of ignorance disguises the natural inequalities that give some people advantages over others. Unfair advantages are removed by depriving the individual of the knowledge that would enable him to identify his self-interest. No one can act upon threats in the original position because no one knows the identity of those who would be harmed or benefited by such acts. For Rawls the state of nature is a condition of primitive equality induced by ignorance about one's personal advantages.

In Hobbes's account all threats are allowed, in Rawls's account no threats are allowed, and in Locke's account most threats of nonparticipation are allowed. Nature as ignorance is the polar opposite of nature as normlessness, and nature as nonparticipation is in between.

F. Conclusion to Part I

We have shown that two leading traditions in political theory can be explicated by two simple mathematical ideas. Specifically, utilitarianism draws upon the idea of a maximum or saddle point, and contractarianism in its economic aspect draws upon the idea of a bargaining equilibrium or fixed point. The two mathematical ideas characterize the pattern of explanation that these political philosophies apply to law.
Political justice consists in the allocation of duties and rights to persons by law and political practice. The two traditions in philosophy which concern us are supposed to explain the allocation of duties and rights to individuals. Having shown how these two philosophies are connected to two mathematical ideas, we now try to connect the mathematical ideas to the structure of law.

II

A recent article on jurisprudence was subtitled "One View of the Cathedral." This image suggests the massiveness of the structure of law and the reverence with which it is approached. Theories are required to make it comprehensible, although they are inevitably inadequate. In Part I we discussed two of the prominent traditions in jurisprudence, namely, utilitarianism and contractarianism. In Part II we shall try to connect these theories to the structure of law.

In order to make this connection it is necessary to reduce the law in all its complexity to a few simple ideas, just as we reduced the two traditions in political theory to simple ideas. We shall not think of the law as a body of living institutions, each with its own personalities and history. Rather, we shall think of the law as a body of rules with a definite logical structure. We conceive of the allocation of duties and rights as being articulated in rules. This reduction is illuminating because the structure of rules has been studied formally by logicians. Our strategy is to connect the logic of rules to the two political philosophies that we have discussed.

A. Preferences and Rules

Two topics in practical logic which have been developed axiomatically are the logic of preferences and the logic of rules. The logic of preferences has been developed by economists under such headings as "the calculus of utilities," "consumer demand theory," or "decision theory." The logic of rules has been developed by philosophers under the heading "deontic logic." We shall explain the connection between these axiomatic systems.

Economic theory defines "preference" as a behavioral relationship between a decision maker and a pair of alternatives. If a person could choose between state one \( s_1 \) or state two \( s_2 \), and he chooses \( s_2 \), then economists say that "\( s_2 \) is revealed to be preferred to \( s_1 \)." In economics, ascribing a preference to someone amounts to predicting the choice that he would make when confronted with a pair of alternatives.

Choices must be made by institutions as well as by individuals. Economists often describe institutional choices as revealing institutional preferences. For example, economists speak of the revealed preferences of a corporation, legislature, or school board.

It is straightforward to extend this mode of speaking to legal systems. Many laws give directions about how to act. We can think of a law as revealing a preference for some actions rather than others. To elucidate, think of an action as changing the state of the world, and think of a law as an instruction to do an action or forbear from changing it. The instruction apparently reveals a preference on the part of the rule maker for the state favored by the action.

The argument can be clarified with the help of the notation used in the logic of rules. The elements of the logic of rules are somewhat different from those of the logic of preferences. We begin with actions, rather than with states of the world. An action changes the world from one state into another. Let \( P_{ij} \) denote the action of changing the state of the world from \( s_i \) to \( s_j \). A rule in canonical form states that certain people ought to undertake a certain action under certain circumstances. Explicit statement of the class of persons and the circumstances is often omitted. For example, \( Op_{ij} \) is the notation for "Someone (unspecified) ought to do the action described by proposition \( p_{ij} \) under certain conditions (unspecified)." It seems reasonable to say that the duty to \( p_{ij} \) "reveals" a preference on the part of the rule maker for \( s_j \) over \( s_i \), which we write

\[ Op_{ij} \iff s_i < s_j. \]

Logicians would say that \( s_j \) is a deontically ideal world relative to an actual world \( s_i \).

B. Contradiction

We connected preferences to rules by using the fact that rules guide actions and actions reveal preferences. Now we wish to connect systems of preferences to systems of rules. In a well-developed logical system, such as consumer-choice theory or deontic logic, there are axioms that characterize consistent reasoning. Violation of these axioms is said to be irrational or contradictory. We shall compare a simple concept of contradiction contained in the logic of preferences to a simple concept of contradiction in the logic of rules.

Let us review part of the argument developed previously. Utili-
Carianism is based upon the idea of maximization, which presupposes a measure of value. A measure of value exists when states of the world can be set in order according to their goodness. Preference relations between pairs of states can be set in order if the preference relations are transitive. For example, if \( s_2 \) is preferred to \( s_1 \), and \( s_1 \) is preferred to \( s_2 \), then transitivity requires that \( s_3 \) is preferred to \( s_2 \). Transitive preferences can be arranged along a line as in Figure 6.

Intransitive preferences cannot be set along a line; they run in a circle. For example, a person who chooses \( s_2 \) over \( s_1 \), \( s_2 \) over \( s_3 \), and \( s_3 \) over \( s_1 \) reveals the circular preferences depicted in Figure 6. Since intransitive choices cannot be represented by a measure of value, they cannot be described as maximizing anything. The failure to maximize is evidence of irrationality according to utilitarian philosophy. Consequently, the basic concept of contradiction in the logic of preferences, and in utilitarian philosophy, is intransitivity.

Figure 6

<table>
<thead>
<tr>
<th>Transitive Preferences</th>
<th>Intransitive Preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 &lt; s_2 &lt; s_3 )</td>
<td>( s_3 \rightarrow s_2 )</td>
</tr>
</tbody>
</table>

If a decision maker is an individual with transitive preferences, then economists call his preferences a utility function. If the decision maker is an institution, then economists often call its transitive preferences a social ordering. The term “social” is used because it connotes a multiplicity of individuals. If a social ordering is consistent with an ethical or a normative ideal, then it is called a social-welfare function. Thus, the federal tax code is said to reveal a social ordering insofar as it is internally consistent. The tax code is said to reveal a social-welfare function insofar as it satisfies an ethical criterion such as basing marginal tax rates on ability to pay.

Now we leave the topic of contradictions among preferences and consider contradictions among rules. We develop the concept of contradictory rules informally at this stage and formally in a subsequent section. We shall distinguish two types of contradiction. The first kind of contradiction involves asserting and denying that a rule exists, i.e., asserting that a certain law is part of the legal order and also asserting that the law is not part of the legal order. For example, it is a contradiction to assert that there is a legal obligation to cross streets only at corners and also that there is no legal obligation to cross streets only at corners. Making both assertions provides no guidance to someone who is wondering whether to cross in the middle of the block. The second kind of contradiction involves asserting that there is an obligation to do an action and also an obligation not to do that action. For example, it is a contradiction to assert that drivers are legally obligated to drive at least 45 MPH on a certain road and also to drive under 45 MPH on that road. Rules which are contradictory in the second sense have the characteristic that obeying one rule violates the other. An ideal system of law would offer a guide to action. Contradictory rules offer no guide to action.

What is the connection between consistency in rules and preferences? One possibility is that a consistent set of rules necessarily reveals a consistent set of preferences. We are entertaining the possibility that consistent rules necessarily reveal transitive preferences. If this assumption were true, then rational laws would necessarily have the mathematical structure of utilitarianism, i.e., the law could be described as maximizing a conception of social value.

Utilitarian jurisprudence is not entitled to such an easy triumph. The fundamental concept of contradiction in rules is weaker than the utilitarian concept. A set of rules can offer an uncontradictory guide to action without revealing an ordered purpose. Rules that are consistent in the sense of offering a consistent guide to action need not be consistent in the sense of revealing a transitive ordering over states of the world.

This argument requires formal proof, which we provide in a subsequent section, but first we shall remove some of its obscurity by discussing a concrete application of it.

C. Distribution, Utility, and Justice

There is a tendency for people to judge whether an economic system is just by examining the distribution of income across classes. Utilitarians hold that that income distribution is best which maximizes the sum of utilities. In practice utilitarians have advocated income redistribution by such means as taxes, subsidies, free necessities (e.g., health care), minimum-wage laws, unionization (monopoly in labor markets), antitrust (no monopoly in product markets), etc. The underlying principle is that more equality is better if it does not cause less production.\(^4\)

For example, in Figure 7 we have drawn a curve showing the levels of utility that can be obtained by tax transfers between persons A and B. The diagram assumes that both people choose how
many hours to work and that A's hourly wage is greater than B's. The tax is levied on the earnings of high-income people such as A and transferred to low-income people such as B. At very high tax rates, neither A nor B works many hours, and both suffer low utility levels, as represented by the points on the utility possibility curve near the origin. By lowering the tax rate we move out to the right on the utility possibility curve, and both people are better off, up to a point. Eventually a point is reached where further decreases in the tax rate cause a decline in subsidy to B. Beyond this point decreases in tax rates benefit A and harm B. Economists call this part of the curve the “Pareto frontier.” The sum of the utilities is maximized at the point of tangency between the Pareto frontier and a straight line sloping down at 45° as shown. This optimal point corresponds to a tax rate that is best according to a utilitarian philosophy. Any other point on the utility possibility curve represents a lower level of utilitarian social welfare than at the optimum.

One way to differentiate utilitarianism from contractarianism is to claim that contractarianism directs society to a different point

Figure 7
Optimal Tax—Transfer

on the Pareto frontier from the utilitarian optimum. For example, the version of contractarianism developed by Rawls concludes that the just point on the Pareto frontier is where the worst-off individual is as well off as possible (maximin in Figure 7).

This way of thinking about income distribution was attacked by Robert Nozick. His interpretation of Locke rejects the whole approach of disputing about the just point on the Pareto frontier. Nozick argued that justice cannot be decided by a “time slice” or “end state” approach? Nozick applies the predicate “just” to a process, not to a still photograph of income classes. He follows the contractarian logic of judging by the history of the world, not by the state of the world.

The object of distributive justice in Nozick's theory is the history of the relationship between people and things, not the pattern of that relationship. Static or time-slice theories apply the judgment about whether something is just to the wrong object. We cannot say that some points on the Pareto frontier are more just than others by looking at the distribution of property across persons; we must know the history of the connection between property and persons. To think otherwise confuses the relationship of people to things. We are not faced with a collection of things to be distributed and enjoyed; rather, we are faced with a collection of things to be transformed by our labor. The process of transformation and transfer is the correct object of our judgment about justice, not the final allocation, according to Nozick.

According to Nozick, any voluntary redistribution from an initially just situation is just, and any involuntary redistribution is unjust. For example, free exchange is voluntary and taxation is involuntary. Thus Nozick's philosophy condemns income taxation, social-security taxation, and minimum-wage laws or other interferences with free contracts.

Nozick's government prefers that contracts be kept, but has no preference over which contracts are made, i.e., no preference over the distribution of property. From the formal viewpoint, the preferences of Nozick's government are created and extinguished as individuals make contracts and carry them out. Nozick's government does not reveal stable preferences over states of the world, because it has no ordered purpose, i.e., no ranking of still photographs of the distribution of property.

A concise illustration of this point is provided by three fundamental laws of an imaginary civil code inspired by Nozick's libertarianism:
1. There is a set of actions called “free actions” which everyone is permitted to do. (Unfree actions are covered by the criminal code.)
2. Everyone is permitted to contract to do any free action.
3. Performing a free action extinguishes the contractual duty to do it.

Notice that this code does not impose civil duties upon anyone except as a consequence of his own voluntary action. If no one creates a contract, then government is indifferent about what free actions are performed. If someone makes a contract, then government prefers that the contract be kept. This preference is extinguished by performance according to the contract’s terms. Rules 1-3 reveal no preference by the rule makers for one state of the world over another. This civil code is an example of a consistent set of laws that does not reveal an ordered purpose. The rule maker cannot be described as maximizing social value. Rather, the rule maker allows maximum freedom to individuals.

We have contrasted a utilitarian tax code with Nozick’s conception of distributive justice. Utilitarians favor intervention in the free exchange of contracts in order to achieve the optimal distribution of property, whereas Nozick holds that such intervention is unjust to individuals. The fundamental point is that utilitarianism imposes a stronger purpose upon the law than some philosophers in the contractarian tradition. We develop this point formally in the next section.

D. Formalization

Some formal notation will clarify our argument, although readers who are allergic to it may wish to skip this section. As before, we let $O p_i$ denote the rule that imposes a duty to do the action described by proposition $p_i$. There are two forms of negation in the logic of rules. Internal negation states that there is an obligation not to do an act $p$, which is written $O \sim p$. For example, there is an obligation for drivers not to pass a school bus that is discharging children. External negation, written $\sim O p$, states that no obligation exists to do the act $p$.

There are two simple concepts of contradiction in the logic of rules corresponding to the two forms of negation. An internal contradiction is the assertion that there is an obligation to do some act and an obligation not to do that act, which is written $O p$ and $O \sim p$.

For example, it is an internal contradiction to assert that there is an obligation to stop for a school bus discharging children and an obligation not to stop for a school bus discharging children. By contrast, it is an external contradiction to assert that there is an obligation to do some action and also no obligation to do that action, which is written $O p$ and $\sim O p$. Contradictory rules offer no guide for action, just as contradictory statements of fact offer no basis for belief.

As before, we let $P_{ij}$ denote the action that changes the world from state $s_i$ to $s_j$. A rule that imposes the duty to do the action $P_{ij}$ is said to reveal a strong preference for $s_j$ over $s_i$:

$$O p_{ij} \iff s_j < s_i.$$ 

Similarly, the duty not to do $P_{ij}$ reveals the opposite preference:

$$\sim O p_{ij} \iff s_i \leq s_j.$$ 

It is a feature of most systems of deontic logic that no obligation to $p$ is identical to permission not to $p$. Consequently, the absence of a duty to $P_{ij}$, or the permission not to $P_{ij}$, seems to reveal that $s_j$ is no better than $s_i$:

$$\sim O p_{ij} \iff s_i \leq s_j.$$ 

This notation enables us to impute preferences to lawmakers. Consider a pair of laws and what they reveal:

$$O p_{12} \iff s_1 < s_2.$$ 
$$O p_{13} \iff s_1 < s_3.$$ 

This pair of duties suggests that lawmakers prefer $s_3$ over $s_2$, and $s_2$ over $s_1$. If the preferences of the lawmakers are transitive, then they also prefer $s_2$ over $s_1$. In this case we would also expect a rule to exist such as $O \sim p_{31} \iff s_2 < s_1$. The set of laws $O p_{12}$, $O p_{23}$ and $O \sim p_{31}$ reveals the transitive preferences $s_1 < s_2 < s_3$.

The rules in the preceding paragraph are consistent and transitive. However, it is easy to write consistent rules that are intransitive, e.g.,

$$O p_{12} \iff s_1 < s_2.$$ 
$$O p_{23} \iff s_2 < s_3.$$ 

or

$$O p_{12} \iff s_1 < s_2.$$ 
$$O p_{23} \iff s_2 < s_3.$$ 
$$O p_{31} \iff s_3 < s_1.$$
These two sets of rules do not contain an internal or external contradiction, but they do not reveal a transitive preference ordering. The conclusion can be generalized:

**Theorem:** Consider a set of laws consisting of duties (e.g., \( Op_{ij} \)) and permissions (e.g., \( \neg O \neg p_{ij} \)). Assume the set of laws are consistent and transitive. Internal or external negation of any duty in a triple of duties will make the laws consistent and intransitive.45

The meaning of this theorem can be expressed in ordinary language. An ideal of legality is that laws should offer intelligible instructions. If one statute imposes a duty to \( p \) and another imposes the duty not to \( p \), then at least one of them is illegal. However, a set of intelligible instructions may not have an ordered purpose, e.g., the instructions may not be transitive. In brief, the set of possible rules that are transitive is a subset of the possible rules that are consistent.

This theorem is potentially significant for the dispute between utilitarians and contractarians. Utilitarians hold that rational laws necessarily have an ordered purpose—namely, maximizing utility—and contractarians deny this claim. If parts of the law can be modeled by systems of deontic logic without the transitivity axiom, then the actual allocation of legal duties and rights must not be utilitarian.

As an illustration consider the Nozickian civil code discussed in the preceding section. We can represent rules 1-3 in our formal notation. Let \( Pp \) denote permission to do the action \( p \) (or equivalently, no obligation not to \( p \), written \( \sim O \sim p \)). Let \( S \) represent the set of free actions or actions not covered by the criminal code. An appealing formalization is the following:

1. \( P_{pij} \) all \( p_{ij} \) in \( S \). (Permission to \( P_{ij} \) for all \( P_{ij} \) in \( S \).)
2. \( PO_{pij} \) all \( p_{ij} \) in \( S \). (Permission to create the duty to \( P_{ij} \) for all \( p_{ij} \) in \( S \).)
3. \( \sim (p_{ij} \text{ and } Op_{ij}) \) all \( p_{ij} \) in \( S \). (There is not both the performance of \( p_{ij} \) and the duty to \( p_{ij} \) for all \( p_{ij} \) in \( S \).)

Rule 1 reveals a weak preference of the form \( s_i \leq s_j \). Rule 2 reveals a weak preference over the action of creating duties, i.e., making contracts. Rule 3 states that any strong preference created by making a contract according to rule 2 can be extinguished. Thus the fundamental rules of the Nozickian civil code do not reveal strong preferences over states of the world. We cannot set states in order of goodness by using the preferences revealed by these laws. Our conclusion is that Nozick's government has no ordered purpose, i.e., there is not a transitivity axiom.

The iteration of operators in rule 2 and the mixed form of rule 3 raise some controversial issues in deontic logic, but exploring them in this essay is inappropriate.44

**F. Implications**

Utilitarian jurisprudence is committed to the view that rational laws serve an ordered purpose, namely, maximizing utility. A set of laws is irrational by utilitarian standards if it does not reveal an ordered purpose. A minimal condition for utilitarian rationality is that obeying the law increases social welfare as measured by a shared standard of value.

Contractarian jurisprudence is not committed to viewing the law as maximizing welfare. Some contractarian philosophies postulate ordered purposes and others do not. Economic contract theory views law as a bargaining equilibrium, which presupposes a function describing how the bargainers react to each other. However, a reaction function does not presuppose a shared concept of value. Economic contractarianism is consistent with the view that the goals of individuals are incommensurable in the sense that they cannot be reduced to a single standard of value.

We have worked out the formal implications of the argument that contractarianism tolerates a weaker purpose in law than utilitarianism. We connected preferences to rules in order to show how the logic of law is different in utilitarian and contractarian models. Utilitarian laws satisfy the transitivity axiom, but contractarian laws are not necessarily transitive. The fundamental laws would not reveal stable preferences over pairs of states, or triples of states, in some versions of contractarianism.45

A set of laws without an ordered purpose may be described as a heterogeneous collection of rights and duties. The basic constitutional liberties are depicted this way by many contractarians. For example, Rawls describes "liberty" as a collection of conventional rights, e.g., freedom of speech, press, religion, association, etc. The value of these liberties is not derived by Rawls from their ability to advance some other purpose, such as maximizing pleasure. The absence of an underlying measure of value makes the comparison of liberty with other values impossible. For example, in Rawls's theory liberty cannot be balanced against economic advantage.46 It is possible to find the set of liberties that afford maximum equal liberty for everyone, but it is not possible in Rawls's system to balance liberties against nonliberties. For Rawls liberty is a heterogeneous collection of rights that are valuable for their own sake.
The idea that a legal system might not be transitive is familiar to economists. Kenneth Arrow's Impossibility Theorem proved that a democratic political constitution cannot combine the preferences of different individuals into a transitive ordering. This conclusion is sometimes expressed by the proposition, "The social-welfare function does not exist." This result has been interpreted as a refutation of utilitarianism and a proof that politics is unavoidably irrational. Our argument shows that some contractarians would not want a constitution with an ordered purpose, even if it were possible to construct one.

SUMMARY

In our opening remarks we asked, "Is justice a sensibility or can it be analyzed mathematically?" Political justice resides in the allocation of rights and duties to individuals by law and political practice. Utilitarianism and contractarianism are two important theories of law. We showed that two simple mathematical ideas can be used to explicate these political philosophies. The concept of a maximum was used to explicate utilitarianism, and the concept of a bargaining equilibrium was used to explicate contractarianism.

The concept of a maximum presupposes an ordered conception of value which guides social choice. Utilitarians view the individuals in society as combining their efforts toward advancing a single end, rather like the individuals in a firm who combine their efforts to maximize profits. A utilitarian allocation of rights and duties maximizes social welfare, which implies that the preferences revealed by such laws satisfy the transitivity axiom.

The concept of a bargaining equilibrium presupposes a reaction function for different individuals, but not a shared conception of value. Economic contractarians view the individuals in society as entering into an agreement to further their private ends, rather like the parties to a business contract. Contractarians do not necessarily view the law as having a unifying purpose such as maximizing social welfare. Contractarians do not believe that law is irrational just because it does not reveal an ordered purpose.

Judgments about justice turn upon subtle distinctions in thought which are buried in the mind. Ethical theories aim at unearthing these distinctions, a kind of archaeology of the mind. We have shown that two simple mathematical ideas can explicate two traditions of political theory. The purpose of this demonstration is to unload the dross accumulated around our intuitions, so that we can think more clearly and build more confidently upon our moral instincts. The danger is that we will use mathematics as a substitute for thinking. If mathematics is used mechanically, then the crucial distinctions are compacted into a worthless conglomerate; such an operation is rather like excavating a Greek temple with a bulldozer. If mathematics is used with discernment and sensitivity, then we can uncover patterns of thought in philosophical traditions which are a precious inheritance.

NOTES

2. Bentham was the first systematic utilitarian. See J.J.C. Smarke, "Utilitarianism," Encyclopedia of Philosophy, for discussion and bibliography.
4. A description of a state of the world is usually defined to include a complete account of causal propensities. As a result, a description of the present state of the world contains a complete prediction of what the future will be if no one acts to change the natural order of events. Arrow writes: "A state of the world is a description of the world so complete that, if true and known, the consequences of every action would be known." Kenneth Arrow, Essays in The Theory of Risk Bearing (Amsterdam: North-Holland, 1971), p. 45.
5. Formally, intransitive preferences are not transitive, and transitive preferences have the property that, for any triple \(s_1, s_2, s_3\), \(s_1\) preferred to \(s_2\), and \(s_3\) preferred to \(s_2\), implies \(s_3\) preferred to \(s_1\).
7. "The root of the whole matter [of contracts] is the reciprocal conventional inducement, each for the other, between consideration and promise." Oliver Wendell Holmes, Jr., The Common Law (Boston: Little, Brown, 1881), pp. 293-94.
9. More generally, these theories assume transferable cardinal utility.
10. The classical text on game theory, which is still the best, is R.
The fixed-point solution discussed in this article is called the Nash equilibrium. The Nash equilibrium describes abstractly the conditions under which bargaining ceases, but not the final distribution of the surplus from cooperation. The other solution concepts attempt to describe the final distribution of the surplus that will be achieved when bargaining ceases. One of the solution concepts describing an exact distribution is the so-called Nash bargaining solution, which we discuss later in this essay. The Nash equilibrium and the Nash bargaining solution should not be confused.

11. Bentham was a prolix classifier. The attributes of utilitarianism listed by Bentham in such classics as The Principles of Morals and Legislation (New York: Hafner, 1973) are numerous and complicated. The list of four attributes used in this essay seems salient in retrospect.

12. “By utility is meant that property in any object, whereby it tends to produce benefit, advantage, pleasure, good, or happiness (all in the present case comes to the same thing) or (what comes again to the same thing) to prevent the happening to mischief, pain, evil, or unhappiness to the party whose interest is considered.” Bentham, The Principles of Morals and Legislation, chap. 1, section IV, p. 2.


15. This is not to deny that future pleasures will be had from memories of past events.


17. “Social Contract is the same given to a group of related and overlapping concepts and traditions in political theory. Like other aggregations in philosophy and intellectual history, it has at its center an extremely simple conceptual model, in this case that the collective is an agreement between the individuals who make it up.” Peter Laslett, “Social Contract,” Encyclopedia of Philosophy, p. 465.


19. This is the traditional view. More recently, Rawls has described the terms of the social contract as the conception of justice in light of which a political constitution is chosen. John Rawls, Theory of Justice (Cambridge, Mass.: Harvard University Press, 1971), pp. 195–201.

20. The economic theory is complicated. In games there may be multiple (Nash) equilibria, only some of which are Pareto efficient. There is no precise statement of the conditions under which bargaining has efficient outcomes. It is sometimes said that bargaining will be efficient if there are no transaction costs. However, the concept of transaction costs has no satisfactory mathematical formulation. This problem is discussed and resolved in Robert Cooter and Stephen Marks, “Bargaining in the Shadow of the Law: Model of Strategic Behavior,” Mimeograph, February 1981.

21. John Rawls argues that the ideal social contract is Pareto efficient but not maximal in the utilitarian sense. See Theory of Justice, p. 70.

22. Contractarians may argue that the social contract directs officials to maximize social welfare under certain circumstances, but the conditions under which this behavior is acceptable are carefully circumscribed in the social contract. Utilitarians may argue that government officials are obligated to keep past commitments, but this obligation is derived from the principle of utility, not from a social contract.

23. Nash’s bargaining solution is described in Luce and Raiffa, pp. 128–34 and 140–43.

24. The principle of distribution advocated by Rawls is to maximize the minimum income. If redistribution does not influence the amount which is produced—i.e., if there is a “lump-sum tax” so that redistribution is costless—then the maximum results in equal shares for everyone, as in our parable in arithmetic.

25. Note that this line of reasoning makes the simplifying and unrealistic assumption that the cooperative product is unaffected by its distribution, i.e., the individuals work just as hard at producing corn in the cooperative venture regardless of how much each consumes. Our parable is too simple to address the problem of incentives.

26. “Out of civil state, there is always war of everyone against everyone.” Thomas Hobbes, Leviathan, ed. Michael Oakeshott, (New York: Collier Books, 1962), p. 100. A contemporary example of a theory with this structure is James Buchanan’s Limits of Liberty (Chicago: University of Chicago Press, 1975). He explains the creation and revision of property rights as an exercise of natural power. Property rights are created or revised whenever the balance of natural power shifts; in Buchanan’s words, the “constitution is re-negotiated.”

27. A contemporary theory that draws upon Locke to defend libertarianism is Robert Nozick’s Anarchy, State and Utopia (New York: Basic Books, 1974).

28. Rawls begins his opus with the sentence, “Justice is the first virtue of social institutions, as truth is of systems of thought.” Theory of Justice, p. 3.

29. It is a feature of our actual legal system that some contracts extracted by threatening nonparticipation are unforeseeable. For example, the law will not enforce a contract which “sells” the cargo of a sinking ship for a trivial sum to the only ship’s captain in a position to save it. (This case is discussed in M. Eisenberg, “Bargain Promise,” mimeo, Berkeley Law School, 1980.) Such examples support Rawls’s position that some free contracts are not fair contracts.
formal logic

A Mathematica/Approach

collection of one of von Wright's atomic changes, or a "molecular agency. So a change in the state of the world, which we called an action, is a


In formal deontic logic the operator for obligation prefixes letters whose exact interpretation is disputed. These letters may stand for act predicates, sentences about acts, or states of affairs. This dispute has little

mean "doing something different." In von Wright's notation, \( \phi \equiv \neg (\neg \phi \rightarrow \phi) \).

In deontic logic each set of norms applicable to a real world is said to presuppose an ideal world in which those norms are not violated.

This utility possibility curve appears in the literature on optimal income taxation. For example, see R. Cooter and E. Helpman, "Optimal Taxation. For example, see R. Cooter and E. Helpman, "Optimal Taxation," Quarterly Journal of Economics, 88 (1974), p. 660.

Nozick, Anarchy, State and Utopia.

Nozick is a libertarian who draws heavily upon Locke's contractarianism. Some contractarians advocate an ordered purpose in the laws governing economic life. For example, Rawls advocates the maximum, whose tax consequences are shown in Figure 6.

von Wright distinguishes between doing an action and forbearing from doing it. We use the term "internal negation" to refer to forbearing from doing an action. However, the phrase "not doing an action" sometimes means something stronger than forbearance. This phrase can also mean "doing something different." In von Wright's notation, \( d \equiv \neg (\neg d \rightarrow d) \).

Our concept of external negation is defined by von Wright as "permission." Specifically, the negation of an obligation to do an action is equivalent to permission to forbear from doing it: \( \neg Op = Pfp. \) Permission to do an action is no obligation not to do it: \( Pd \equiv \neg Op = \neg Op \equiv \neg Op. \) See von Wright, p. 140.

A set of obligations is consistent only if all obligations in this set can be simultaneously fulfilled and each permission can be realized without violating any obligation. Hilpinen, p. 16. Obviously, the obligations \( Op \) and \( O \rightarrow p \) cannot be fulfilled simultaneously. A permission is no obligation not to do the act in question: \( Pp \equiv \neg Op \rightarrow p. \) Consequently, we have \( (Op \land \neg Op) = (Op \land P \rightarrow p). \)

The permission cannot be acted upon without violating the obligation. We have shown that our concept of internal contradiction corresponds to two duties that cannot be simultaneously performed, and our concept of external contradiction corresponds to a permission that cannot be realized without violating an obligation. In a more formal approach, there is no simple concept of contradiction; rather there is a set of axioms whose violation is a contradiction of the system.

It is possible to construct a system of deontic logic without an axiom prohibiting what I call "internal contradiction." See John Lemmon, "Moral Dilemmas," Philosophy Review, April 1962, p. 139. The connection to morality is discussed in Ruth Marcus, "Moral Dilemmas and Consistency," Journal of Philosophy, March 1980, p. 121, and the application to law is discussed in Stephen Munzer, "Validity and Legal Conflict," Yale Law Journal, May 1973, p. 1140. I attempt to avoid this dispute by claiming that consistency, or the absence of dilemmas, is an ideal of legality, although not necessarily a characteristic of law.

Sketch of a proof by construction: Consider any duty \( Op \).

reveals a strict preference \( s_j > s_i \). Internal negation reverses the strict preference, yielding \( s_j < s_i \), and external negation weakly reverses the strict preference, yielding \( s_j \geq s_i \). Either change results in intransitivity when \( Op \) is embedded in a triple of strict duties, but neither change results in a deontic inconsistency.

For example, it is easy to prove that rules 1 and 3 imply \( (Pp \land P \rightarrow p) = Pp \land P \rightarrow p \) is possible. This result is unproblematic if we read the conclusion as "It is possible that doing \( p \) is permitted and undoing \( p \) is permitted," or "It is possible to do \( p \) and not-do \( p \) in such a way that no duty is violated and to not-do \( p \) in such a way that no duty is violated. However, the conclusion is problematic if it is read "Everything is permitted."

Liberty is "lexically prior" to economic advantage in Theory of Justice, p. 61.