The formulation of a utilitarian tax philosophy in the nineteenth century by John Stuart Mill and others did not lead to the conclusion that there should be a progressive income tax. That conclusion was reached by Francis Edgeworth when he demonstrated that equal marginal sacrifice requires equal after-tax income for different individuals in the absence of "announcement effects" from taxation. The proof established a presumption that a utilitarian tax philosophy requires progressive income taxation, which lies behind contemporary ability-to-pay arguments for progressivity.

Edgeworth recognized the important effects of income taxation upon work effort, but he was unable to incorporate these effects explicitly into his mathematics. Recent formulations have corrected this omission and a revision of utilitarian tax philosophy is underway which is as fundamental as what took place after Edgeworth first obtained his results. The pioneer in this work is James Mirrlees. Unfortunately, much of the theoretical literature is technical and abstruse. This paper derives the fundamental theorems by methods which are intelligible to anyone familiar with the Maximum Principle.

I. Optimal Income Taxation with Two Goods

The original papers on optimal income taxation concerned the effect of taxation upon work effort or investment in education. My exposition will be based upon the effort-incentive model, in which the individual responds to the tax by adjusting his labor supply. The individual's utility function is written

\[ u = u(-y, x) \]

where \( x \) is consumption, \( y \) is labor, and \( u \) is assumed to be concave and twice differentiable. Individuals differ with respect to their productive skill \( n \), which is assumed to be equal to their gross wage rate. Total wages before taxation for an "\( n \)-person" are the product of his wage and labor supply: \( z = yn \). Consequently, we may write the problem of individual choice by an \( n \)-person

\[
\max_{z, x} u(-z/n, x) \\
\text{subject to } z - T(z) - qx \geq 0
\]

where \( T(\cdot) \) is the income tax schedule; \( q \) is the price of \( x \), which may as well be unity in the two-good model. I shall assume an interior solution, so the first-order conditions are

\[
\begin{align*}
\frac{\partial u}{\partial z} &= n(1 - T'(\cdot))
\Psi \\
\frac{\partial u}{\partial x} &= \Psi \\
z - T(z) - x &= 0
\end{align*}
\]
Note that $\Psi$ is the Lagrange multiplier which indicates the marginal utility of expenditure. Subscripts on functions indicate partial derivatives.

A feature of this model is that an individual's productive skill or hourly wage is assumed to be independent of the tax structure; consequently, there is a distribution $f(n)$ which is unaffected by the control variable. I also assume $f(n)$ to be continuous. Let $x_n$ and $y_n$ indicate the consumption and labor of an $n$-person. The subscripts serve to remind us that the variables depend upon the person's productive skill; I omit these subscripts in the text to avoid cluttered notation. The government's problem is to maximize an additive social welfare function $G(u(\cdot))$ subject to the government's revenue need $R$ and the labor-supply response of individuals:

$$\max_{z,x,T} \int_{N_1}^{N_2} G(u(-z/n,x)) f(n) \, dn$$

subject to

$$R = \int_{N_1}^{N_2} T(z) f(n) \, dn$$

$$T(z) = z - x$$

For convenience we take $N_1$ and $N_2$ to be nonnegative and finite. The government chooses the tax/subsidy schedule $T(\cdot)$; it also chooses the consumption schedule $x_n$ and the income (hence labor supply) schedule $z_n$, but choice is constrained by the individual maximizing conditions (1).

We may use the Maximum Principle to solve this problem if we reformulate the constraints as differential equations. First we combine (3) and (4); and then differentiate the combined equation so that the government's budget constraint can be written as a differential equation and a terminal condition:

$$DR = (z - x) f(n)$$

$$R_{N_2} = R$$

(The symbol $D$ is used throughout for the derivative $d/dn$.) I substitute (5) and (6) for (3) and (4) in the maximization problem; by this step I eliminate one control variable, namely $T(\cdot)$, and add a state variable, namely $R_n$. Of course $T(\cdot)$ is still chosen implicitly since it is the difference between income and consumption for each person.

The next step is to write the individual maximizing conditions (1) as a differential equation. Define the utility of an $n$-person when $z$ and $x$ are optimally chosen to satisfy (1):

$$v_n = \max_{u(-z/n,x)} G(u(-z/n,x))$$

By differentiating with respect to $n$ we obtain

$$Dv = u_1 Dz/n + u_2 Dx + u_1 z/n^2$$

I wish to simplify (8). Differentiate the individual's budget constraint with respect to $n$ and substitute the other two conditions from (1) into it, which leaves the equation

$$-u_1 Dz/n + u_2 Dx = 0$$

In view of this fact, (8) may be simplified.

$$Dv = u_1 z/n^2$$

Equation (1) implies (10); it will be assumed that the converse is also true. The special circumstances under which this equivalence does not hold are a technical detail which is relegated to a footnote. By this assumption we may substitute the constraints (7) and (10) for (1). In fact we may simplify further by inverting (7) $z = g(v,x,n)$. ($v_n$ is monotonically decreasing in $z_n$, so inversion is permitted.) This inversion enables us to eliminate $z_n$ as a control and use $v_n$ as a state variable. The final form of the problem is

$$\max_{x,v,R} \int_{N_1}^{N_2} G(v) f(n) \, dn$$

subject to

$$v_n = \max_{u(-z/n,x)} G(u(-z/n,x))$$

(7)

By differentiating with respect to $n$ we obtain

(8) $Dv = u_1 Dz/n + u_2 Dx + u_1 z/n^2$

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$$\max_{x,v,R} \int_{N_1}^{N_2} G(v) f(n) \, dn$$

subject to

(7) $v_n = \max_{u(-z/n,x)} G(u(-z/n,x))$

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$$\max_{x,v,R} \int_{N_1}^{N_2} G(v) f(n) \, dn$$

subject to

(7) $v_n = \max_{u(-z/n,x)} G(u(-z/n,x))$

(8) $Dv = u_1 Dz/n + u_2 Dx + u_1 z/n^2$

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$$\max_{x,v,R} \int_{N_1}^{N_2} G(v) f(n) \, dn$$

subject to

(7) $v_n = \max_{u(-z/n,x)} G(u(-z/n,x))$

(8) $Dv = u_1 Dz/n + u_2 Dx + u_1 z/n^2$

I wish to simplify (8). Differentiate the individual’s budget constraint with respect to $n$ and substitute the other two conditions from (1) into it, which leaves the equation

(9) $-u_1 Dz/n + u_2 Dx = 0$

In view of this fact, (8) may be simplified.

(10) $Dv = u_1 z/n^2$
\[ Dv = u_1(-g(v, x, n)/n, x) \cdot g(v, x, n)/n^2 \]
\[ DR = (g(v, x, n)-x)f(n) \]

with the terminal condition \( R_{N_2} = \bar{R} \) and the initial condition \( R_{N_1} = 0 \), where \( x_n \) is the control and the state variables are \( v_n \) and \( R_n \); by straightforward application of the Maximum Principle we obtain the Hamiltonian and first-order conditions:

\[ H = G(v)f(n) + \mu u_1(\cdot)g(\cdot)/n^2 \]
\[ + \lambda(g(\cdot) - x)f(n) \]

I: \( \partial H/\partial x_n = 0 \)

II: \( \partial H/\partial v_n = -D\mu \)

III: \( \partial H/\partial R_n = -D\lambda \)

IV: \( Dv_n = u_1 \cdot g(\cdot)/n^2 \)

V: \( DR_n = (g(\cdot)-x)f(n) \)

VI: \( \mu_{N_1} = \mu_{N_2} = 0 \)

(transversality conditions)

VII: \( R_{N_1} = 0; R_{N_2} = \bar{R} \)

All of the theorems are obtained by interpretation of the necessary conditions I–VII.

THEOREM 1: A person with higher productive skill enjoys at least as high utility as a person with lower productive skill; formally, \( Dv_n \geq 0 \).

PROOF:

The theorem follows directly by interpreting the signs in condition IV. This first theorem reflects the fact that a person with higher ability can always earn the same income and pay the same taxes as someone with lower ability, but enjoy greater leisure. Theorem 1 is the motivation for Figure 1 in which equal utilities is only achieved at the origin by tax rates which are so high that neither person works. Figure 1 illustrates a limitation of income taxation, namely that the point of equal utility for everyone is off the Pareto frontier.

THEOREM 2: The marginal tax rate on income is less than one; \( T' < 1 \).

PROOF: (i) Expand condition I to obtain \( T' = 1 - u_1/\Psi n \)

In an income interval where \( T' > 1 \), an increase in gross income produced by a sacrifice of leisure would result in a decrease in income net of tax. No one would exchange labor for a decrease in income, so no one's preferred \( z \) will fall inside this interval. Either everyone will choose \( z \) below that interval, in which case the marginal tax in the interval may as well be reduced, or else someone's utility function will have two local maxima, in which case there is a non-convexity. The theorem tells us more about the limiting assumptions of the model than about reality.

THEOREM 3: The marginal tax rate is nil for the person of lowest ability and for the person of highest ability; \( T'(z_{N_1}) = T'(z_{N_2}) = 0 \).

PROOF: (i) Expand condition I to obtain

8The original observation that marginal taxes are nil at the upper end was made by Efraim Sadka and Edmund Phelps; the observation that marginal taxes are nil at the lower end is attributed to Seade (1975).
\[ 0 = \frac{\partial H}{\partial x} = \mu \left\{ \frac{\partial}{\partial x} \left( u_1 g/n^2 \right) \right\} + \lambda (g_2 - 1) f \]

(ii) Observe that
\[ g_2 = \frac{dz}{dx} = \text{marginal rate of substitution} \]
\[ = +1/(1 - T') \] from the individual maximizing conditions (I)

(iii) Combine (i) and (ii) to obtain the marginal income tax rate:
\[ T' = \frac{-\mu_1 f}{\lambda f g_2} \]

(iv) The theorem follows immediately from (iii) and the transversality conditions VI
\[ \mu_{N_1} = \mu_{N_2} = 0 \]

The transversality conditions require the shadow price \( \mu \) to be nil at the upper and lower ends of the integral because the state variable \( v_c \) is unconstrained by initial or terminal conditions. (At the optimum the vector of shadow prices must be orthogonal to the tangent plane on the manifold representing the initial or terminal constraints.)

There is a simple interpretation for the conclusion that marginal tax rates should be nil for the person of greatest productive skill. Consider any tax schedule with \( T'(z_{N_2}) > 0 \); now construct another tax schedule identical to the first for all \( z \prec z_{N_2} \) and set the right derivative \( T'(z_{N_2}) = 0 \), as shown in Figure 2. Tax collections from every individual are the same under both schedules, so no one of lower ability is worse off. However, the person of highest ability has a larger opportunity set under the second schedule and he may choose to work more; since his welfare is improved, the original schedule cannot have been optimal under the Pareto criterion or any social welfare function which gives positive weight to his interests.

Proving that the tax schedule is non-decreasing is difficult in spite of the triviality of the result. The following theorem is proved in the Appendix:

**THEOREM 4:** The optimal income tax schedule is nondecreasing \( (T' \geq 0) \) if the following conditions are met:

(i) Cross partial is nonnegation: \( u_{12} \geq 0 \)

(ii) The marginal social value of leisure decreases with ability: \( d(G'u_1)/dn \leq 0 \)

If one person enjoys a higher hourly wage than another, then he enjoys higher utility according to Theorem 1. An implication of assumption (i) is higher consumption increases the value of leisure; assumption (ii) requires the tax system to assign less weight to his leisure. In other words, assumption (ii) requires the tax system to distribute leisure in a way which offsets unequal consumption, which is why the assumption is appealing.

From Theorems 3 and 4 we can conclude that there is a zone of increasing marginal tax rates and a zone of decreasing marginal rates; they do not increase everywhere as might be supposed from the Edgeworth-Pigou tradition. Furthermore, the zone of decreasing marginal rates occurs at a higher income level than the zone of increasing rates. If the marginal tax rate for the optimal schedule has a single maximum, then the schedule of marginal rates is concave,

9The role of the cross partial's sign in determining the nonnegativity of the marginal tax rate is discussed by Sadka.
with an interior maximum. Of course the theorems tell us nothing about the rate at which the optimal marginal rates approach zero; simulations must determine whether Theorem 3 is useful or a curiosity. What conclusions can be drawn about progressivity? Progressivity means that an increasing proportion of income is paid in taxes; we can conclude nothing about progressivity from the theorems because they refer to marginal rates only and say nothing about the intercept (tax on zero income). It seems reasonable from simulations (and can be proven under particular circumstances) that the tax rate on zero income will be negative; a demogrant will be paid to everyone.\textsuperscript{10} The theorems say nothing about its size so I can say nothing about average tax rates.

The two-commodity model captures the disincentive effects of income taxation upon work effort. An obvious criticism of this model is that it tells nothing about a general system of taxation; I shall remedy that in the next section.

II. Tax Schedules for Many Goods

It is administratively possible to have a tax schedule for goods other than income. For example housing is subsidized (negative tax) for some people at a rate which varies with their expenditure upon it. In this section \( x \) is interpreted as a vector of goods, each of which is susceptible to a tax schedule. However, it is unrealistic to assume that a tax schedule is administratively feasible for every commodity; for most commodities government must be content with tax rates, particularly when resale is possible. Let \( c = (c_1, c_2, \ldots, c_m) \) be a vector of commodities which are susceptible to tax rates but not tax schedules. Mirrlees' problem is to optimize the tax schedules for \( z \) and \( x \), and Ramsey's problem is to optimize the tax rates for \( c \). I set up the control problem and solve Mirrlees' problem in this section; in the next section Ramsey's problem will be solved.

I wish to maximize the weighted sum of utilities subject to the government's budget constraint and the conditions for individual utility maximization. As in Section I, the key to the problem is formulating the constraints as differential equations. First consider the problem of individual utility maximization. In the general formulation an \( n \)-person must solve

\[
\max_{z,x,c} u(-z/n, x, c)
\]

subject to

\[
-z + T(z) + \sum_i (p_i x_i + Q^i(x_i)) + \sum_j q_j c_j = 0
\]

\[
\equiv m
\]

where

- \( z \) = labor income
- \( T \) = income tax
- \( p = (p_1, p_2, \ldots) \) , seller’s prices
- \( x = (x_1, x_2, \ldots) \) , a vector of consumption goods
- \( Q^i \) = tax schedule for \( x_i \)
- \( q = (q_1, q_2, \ldots) \) , buyer’s prices
- \( c = (c_1, c_2, \ldots) \) , a vector of consumption goods
- \( m \) = net expenditure

Equation (11) can be expressed as a differential equation if we follow the same tack as the two-good case. Define utility as a function of the optimally chosen commodity bundle (satisfies (11)):

\[
v_n = \max u(-z/n, x, c)
\]

Differentiate this function and the budget constraint in (11) with respect to \( n \); combine these results with the first-order conditions in (11) to obtain our familiar differential equation:

\[
Dv = \frac{\partial}{\partial n} u(-z/n, x, c) = u_z z/n^2
\]

Notice that (11) implies (13); it will be assumed that the converse is also true. (This assumption was discussed for the two-good case in fn. 7.)
It is customary to set up Ramsey’s problem by using the indirect utility function rather than the direct utility function, because the government chooses consumer prices rather than quantities. Similarly, we shall use the expenditure function so that we can make the vector of buyer’s prices \( q \) into the control variable, rather than the quantities \( c \).

Consider any system of tax schedules \( T \) and \( Q \). What is the minimum expenditure which realizes a particular utility level \( v \), given \( z/n, x, \) and \( q \)?

\[
\min_c -z + T(z) + \sum_i [p_i x_i + Q_i(x_i)] + \sum_j q_j c_j
\]

subject to \( u(-z/n, x, c) \geq v \)

Let the solution be the expenditure function

\[
m = m(q, v, -z/n, x)
\]

with the partial derivatives being Hicksian demand

\[
m_j(\cdot) = c_j
\]

By substitution into (13) we can eliminate \( c \) from the differential equation,

\[
Dv = (z/n^2)u_1(-z/n, x, m_q)
\]

The differential equation we have been seeking is (17).

We can write the revenue constraint as a differential equation and a terminal condition just as in the two-good case. The final formulation of our general problem is written

\[
\max_{z, x, v, R, q} \int_{N_1}^{N_2} G(v) f(n) \, dn
\]

subject to

\[
Dv = (z/n^2)u_1(-z/n, x, m_q(\cdot))
\]

\[
DR = (z - \sum_i p_i x_i - \sum_j p_j m_j(\cdot)) f
\]

and the conditions

\[
R_{N_1} = 0
\]

\[
R_{N_2} = \bar{R}
\]

The controls are \( z, x, \) and \( q \); the state variables are \( v \) and \( R \). I find the following first-order conditions by application of the Maximum Principle:

\[
H = G(v) f(n) + \mu \frac{\partial u}{\partial n} (\cdot)
\]

\[
+ \lambda(z - \sum_i p_i x_i - \sum_j p_j m_j(\cdot)) f
\]

I: \( 0 = \frac{\partial H}{\partial z} = \frac{\partial H}{\partial x} \)

II: \( -D\mu = \frac{\partial H}{\partial v} \)

III: \( -D\lambda = \frac{\partial H}{\partial R} \)

IV: \( Dv = \frac{\partial u}{\partial n} \)

V: \( DR = (z - \sum_i p_i x_i - \sum_j p_j m_j) f \)

VI: \( \mu(N_1) = \mu(N_2) = 0 \)

VII: \( R_{N_1} = 0; R_{N_2} = \bar{R} \)

Different values may be chosen for \( z, x, v, \) and \( R \) at different values of \( n \); the vector of prices \( q \) may also be chosen, but \( q \) is invariant with respect to \( n \). The first-order condition on the choice of the \( q \)'s is obtained by the usual reasoning in the calculus.

VIII: \( 0 = \frac{\partial J}{\partial q_k} \) all \( k \)

where \( J = \int_{N_1}^{N_2} \{ G(v) f(n) + \mu(n)(-Dv + \frac{\partial u}{\partial n}) + \lambda(-DR) + (z - \sum_i p_i x_i - \sum_j p_j m_j) f(n) \} \, dn \)

Theorems 1 and 2 and their proofs carry over from the two-good case without any change, so there is no need to repeat them. Theorem 3 is revised slightly:

THEOREM 3': The change in direct and indirect tax liability that results from a marginal increase in labor income (and hence a marginal increase in expenditure) is nil for the person of lowest ability and highest ability; formally, if \( n = N_1 \) or \( N_2 \), then
where \( T = q - p. \) (See the Appendix for proof.)

The reader will recall from the heuristic explanation in Section I that it is inefficient to discourage these people from supplying an additional unit of labor, which is why they face zero marginal tax rates. In the multicommodity case, this means that the marginal influence of direct and indirect taxation must be nil.

An implication of Theorem 3' is that the direct tax liability from a marginal increase in labor income for the person of lowest or highest ability must be negative if the indirect liability is positive. In other words, positive commodity taxes at the optimum imply a subsidy on marginal income from labor for the worst-off person and the best-off person. In view of this observation, I cannot expect to prove that the optimal marginal income tax is nonnegative everywhere; instead I must frame our nonnegativity proposition in terms of the total tax liability from marginal earnings:

**THEOREM 4':** The change in direct and indirect tax liability, \( \frac{\partial \Delta \tau}{\partial z} \), that results from a marginal increase in use of any of these commodities is nil; formally, if \( u_{1k} = 0 \) all \( k \), then

\[
\frac{\partial}{\partial x_k} \left( T(z) + \sum_i Q'(x_i) + \sum_j \tau_j m_j(\cdot) \right) = 0
\]

(See the Appendix for proof.)

This theorem implies, for example, that there should be no tax/subsidy on housing if there is a tax/subsidy on labor income. This result is not surprising once we recognize that the path of the state variable \( v \), independent of the \( x \)'s in the separable case by condition IV. Control over the path of utilities is achieved by manipulation of \( z \); manipulation of the \( x \)'s adds nothing.

### III. Tax Rates for Many Commodities

I shall review Ramsey's problem and his classical conclusions before generalizing them. The Ramsey problem is to choose commodity taxes which will meet the government's revenue need at minimum loss of utility to a single, representative consumer. There are various statements of the solution; I shall offer four of them after a comment on notation.

The notation is consistent throughout the paper; \( \Psi \) is the marginal utility of expenditure and \( \lambda \) is the shadow price on government revenue. The ratio \( \Psi / \lambda \) converts private expenditure into units which are comparable to government revenue, which is taken as the unit of account. So it is useful to introduce the symbol \( w = 1 - \Psi / \lambda \), which is the deviation of the social value of private expenditure from its nominal value. Thus \( 1 - w \) is the social value of private expenditure.

Four versions of Ramsey's rules are as follows:

**PROPOSITION 1:** (Cost = Benefit) The social cost of the reduction in private expenditure from a marginal increase in the tax liability that results from a marginal increase in use of any of these commodities is nil; formally, if \( u_{1k} = 0 \) all \( k \), then

\[
\frac{\partial}{\partial x_k} \left( T(z) + \sum_i Q'(x_i) + \sum_j \tau_j m_j(\cdot) \right) = 0
\]

(See the Appendix for proof.)
rate on any commodity equals the resulting increase in government revenue; formally

\[(1 - w)m_k(\cdot) = \frac{\partial R}{\partial \tau_k} \quad \text{all } k\]

**Proposition 2:** (Diamond and Mirrlees) The ratio of the tax revenue from a marginal increase in the tax on any commodity to the quantity of that commodity is constant; formally

\[\frac{\partial R}{\partial \tau_k} = (1 - w) \quad \text{all } k\]

**Proposition 3:** (Ramsey) The optimal set of commodity taxes reduces the consumption of every commodity by the same proportion; formally

\[\sum_j \tau m_{jk}(\cdot)/c_k = \sum_j \frac{\partial c_j}{\partial m} - w \quad \text{all } k\]

where \(m_{jk}(\cdot)\) is the Slutsky substitution term. (Notice that the right side of the equation is independent of \(k\).)

**Proposition 4:** (Ramsey) When cross-price elasticities are nil, the optimal tax on any good is inversely proportional to its elasticity of demand; letting \(\theta_k\) be the tax on value \((\theta_k p_k = \tau_k)\),

\[\theta_k = w/\eta_k \quad \text{all } k\]

where \(\eta_k\) is the demand elasticity (defined to be positive).

There are two differences between Ramsey's rules and the propositions which I am about to derive from the general formulation. First, in Ramsey's problem the price of one commodity is varied while holding other commodity prices constant; in the general problem one control is varied while holding the others constant, which includes \(z\) and \(x\). Some of the price effects which I shall examine do not permit a general equilibrium response by the consumer; rather he is constrained to respond by adjusting only his consumption of \(c_1, c_2, \ldots\). In particular the price elasticity of government revenue in the following theorems differs from measured elasticity to the extent that consumers respond to changes in tax rates by adjusting demand for goods subject to tax schedules.

Second, in Ramsey's problem the social value of private expenditure \((1 - w)\) is the same on every good, but it is different for each good in the general problem. It is the same in Ramsey's problem because there is only one consumer and he equates the marginal utility of expenditure on different commodities in order to maximize utility. It is different in the general setting because different consumers buy the same good in different quantities and the marginal social utility of expenditure is different for different consumers. In brief the general problem must take account of the distributional consequences of commodity taxes.

Additional notation is required. Define \(\bar{m}_k\) to be the nominal cost (compensating variation) of a small increase in the price of commodity \(c_k\):

\[\bar{m}_k = \int_{N_1}^{N_2} m_k f dn\]

The social cost is obtained from the nominal cost by converting private expenditure into units comparable to government revenue and taking account of the distribution effect: thus \(-\mu \Psi/\lambda\) is the marginal rate of substitution between \(m\) and \(R\). It is the shadow price of an \(n\)-person's expenditure when government revenue is the numeraire. In addition \(m_n\) is the labor supply of an \(n\)-person, or, if you will, the private worth of the gross wage; the distribution effect is captured by \(m_{kn}\), which is the effect of the price increase upon the private worth of the gross wage. Now define \(w_k\):

\[w_k = -\int_{N_1}^{N_2} \frac{\mu \Psi}{\lambda} m_{kn} dn/\bar{m}_k\]

The social value of private expenditure \((1 - w)\) figured prominently in Propositions 1–4; in their generalization it is replaced by \(1 - w_k\), which is the average social value of expenditure on \(c_k\).

The generalizations of Ramsey's results are obtained from condition VIII:

\[0 = \frac{\partial F}{\partial q_k} \quad \text{all } k\]

The theorems are shown in the text; the proofs are in the Appendix.
THEOREM 6a: \((\text{Cost} = \text{Benefit})\) The social cost of the reduction in private expenditure from a marginal increase in the tax rate on any commodity equals the resulting increase in government revenue from commodities \(c_1, c_2, \ldots\); formally

\[ (1 - w_k)\bar{m}_k = \left. \frac{\partial R}{\partial \tau_k} \right|_{x_k} \quad \text{all } k \]

THEOREM 6b: The ratio of the tax revenue from goods \(c_1, c_2, \ldots\) caused by a marginal increase in the tax on any commodity, to the weighted quantity of that commodity is a constant; the weight is the average social value of private expenditure on the commodity. Formally

\[ \frac{\partial R}{\partial \tau_k} \left|_{x_k} \right. = 1 \quad \text{all } k \]

where \(\overline{c}_k\) is total consumption of commodity \(k\).

THEOREM 6c: The optimal set of commodity taxes reduces the consumption of good \(k\) by a proportion which is decreasing in the average social value of private expenditure on that good; consumption of all goods are reduced in the same proportion if the average social value of private expenditure is the same for each good. Formally,

\[ \int_1^n \sum_{j} \tau_j m_{jk} \, dfn = \frac{1}{\bar{c}_k} = w_k \quad \text{all } k \]

THEOREM 6d: When cross-price elasticities are nil within the commodity group \(c_1, c_2, \ldots\), the optimal commodity tax is inversely proportional to the "average" demand elasticity and decreasing in the average social value of private expenditure on the commodity; formally

\[ \theta_k = \frac{w_k}{\eta} \quad \text{all } k \]

where \(\theta_k p_k \equiv \tau_k\)

\[ \bar{\eta} = - \int_1^n \frac{q_k \frac{\partial c_k}{\partial q_k}}{\bar{c}_k} \, dfn \]

We can obtain corollaries to our theorems by using the following facts, which are derived from the definition of \(w_k\):

(i) \(m_{nk} = 0\) \(\Rightarrow (w_k = 0)\)

(ii) \((\mu < 0 \& m_{nk} > 0) \Rightarrow (w_k > 0)\)

(iii) \((\mu < 0 \& m_{nk} < 0) \Rightarrow (w_k < 0)\)

For example, we obtain the following corollary by plugging these facts into Theorem 6d:

COROLLARY: Assume that cross-price elasticities are nil within the commodity group \(c_1, c_2, \ldots\). The optimal tax rate on \(c_k\) is

(i) nil if the consumption of \(c_k\) remains constant when earning ability \(n\) increases,

(ii) positive when consumption of \(c_k\) increases with earning ability \(n\) and we have the standard case where \(\mu \leq 0\), and

(iii) negative when consumption of \(c_k\) decreases with earning ability \(n\) and we have the standard case where \(\mu \leq 0\).

(i) tells us that tax rates are inferior policy tools when their use has no distributional impact; we should use tax schedules. (ii) and (iii) tell us that commodity taxes should reduce differences in utility levels.

IV. Concluding Remarks

An objection to the optimal income tax literature is that it tells nothing about a general system of taxation. We have seen how to remedy this complaint by deriving the major theorems in a setting with many persons and many commodities. Intuitions about tax schedules obtained from the Edgeworth-Pigou tradition are misleading; that tradition leads us to expect that marginal tax liability from labor income will rise everywhere, but we find that it rises at first and later falls. The conclusion is independent of the particular value of the elasticity of labor supply, provided that its sign is not perverse or zero. The Ramsey results on commodity tax rates hold up in the general setting after adjustments are made for distributional effects of expenditure on different goods.

There remains another objection to this literature, namely that it incorporates only one kind of incentive effect from taxation; a large object is balancing precariously on a small pedestal. In particular it tells us nothing about the incentive effects from taxation.
of income from capital. This criticism misses the mark insofar as income from capital is obtained in proportion to an expenditure of effort, as with human capital. It also misses the mark if capital accumulation is a consequence of saving, since the elements of $x$ may be dated commodities. However, rapid capital accumulation by an individual is typically a consequence of superior information, either about techniques (innovation) or markets (insiders). Our model does not capture the effects of taxation upon the creation and distribution of information; this reflects the unsatisfactory state or absence of a general theory of an economy in which information is costly.

**APPENDIX**

**Lemma used in proving Theorem 4:**

If $d(G'u_1)/dn \leq 0$, as assumed in Theorem 4, then $\mu \leq 0$.

**PROOF:**

(i) By II we have

$$-D\mu(n) = \left( (G' + \lambda g_1(\cdot)) f \right) + \left[ \frac{\partial H}{\partial \lambda} \left( \frac{\partial u}{\partial n} \right) \right] \mu;$$

define $a$ and $b$ so that

$$= [a(n)] + [b(n)]\mu$$

(ii) Solve the linear differential equation:

$$-\mu(n^*) = \int_{n_1}^{n_2} a(m) \exp\left\{ \int_{n_1}^{m} b(\tilde{m})d\tilde{m} \right\} dm$$

The exponential function is nonnegative, so the sign at each point along the integral is given by sign of $a(m)$.

(iii) $g_1(\cdot) = -n/u_1$ by expanding the derivative

$$\therefore a(m) = \left( G' \frac{u_1}{m} - \lambda \right) \left( \frac{fm}{u_1} \right)$$

$$a(m) \geq 0 \iff (G'u_1/n \geq \lambda)$$

(iv) From (iii), the assumption that $G'u_1$ is monotonic decreasing, and the fact that $\lambda$ is constant, I conclude that $a(m)$ changes sign no more than once and any change is from $+\to-$. The conclusion follows from (ii) and the transversality conditions $0 = \mu(N_1) = \mu(N_2)$.

**THEOREM 4:** $T' \geq 0$ if

(i) $u_{12} \geq 0$

(ii) $d(G'u_1)/dn \leq 0$

**PROOF:**

(i) From proof of Theorem 3 we have

$$0 = \frac{\partial H}{\partial x} = \mu \frac{\partial}{\partial x} [u_1 g/n^2] + \lambda (g_2 - 1) f$$

and $g_2 = \frac{1}{1 - T'}$. 

$$\therefore \frac{1}{1 - T'} - 1 = \left( \frac{-\mu}{\lambda f} \right) \frac{\partial}{\partial x} \left[ u_1 \cdot g/n^2 \right]$$

(ii) $\mu \leq 0$ by preceding Lemma. $\lambda$ is a constant by III; we can see that it is positive by complementary slackness.

(iii) We need only prove that $\partial[u_1 \cdot g/n^2]/\partial x > 0$ to establish the theorem. We proceed by differentiating:

$$\frac{\partial}{\partial x} (u_1 g/n^2) = \frac{-gg_2 u_{11}/n + gu_{12} + g_n u_1}{n^2}$$

We know that $u_{11} < 0$ by concavity. Also $g_x$ is the marginal rate of substitution between $z$ and $x$, which is positive. The conclusion follows from the assumption $u_{12} \geq 0$.

**Comment:** Obviously there are weaker conditions under which $T' \geq 0$. It is sufficient if $u_{12} \geq g_x u_{11}/n - g_x u_{12}/g$. I have used the condition on the cross partial in the theorem because it is intelligible, not because it is general.

**Lemma used in proving Theorem 4':**

If $d(G'\Psi)/dv \leq 0$ as assumed in Theorem 4', then $\mu \leq 0$.

**PROOF:**

(i) $-D\mu = \left[ (G' - \lambda \sum_j p_j m_j) f \right] + \left[ \frac{\partial}{\partial v_n} \left( \frac{\partial u(\cdot)}{\partial n} \right) \right] \mu$ by Condition II; define $a$ and $b$

$$= [a(n)] + [b(n)]\mu$$
(ii) \[ \sum_j p_j m_j(\cdot) = \sum_j \frac{p_j}{u_j} \]
by the individual maximizing conditions (11)
\[ \therefore a(m) = (G'\Psi - \lambda \sum_j \frac{p_j}{q_j} f) \frac{f}{\Psi} \]

Notice that \[ \sum_j \frac{p_j}{q_j} \]
is constant as \( m \) varies.

(iii) The proof is completed by following steps (ii)-(iv) in the proof in the preceding Lemma (used in proving Theorem 4), recalling that \( v \) is increasing in \( n \) by Theorem 1 so that \( (dG'\Psi/dv \leq 0) \Rightarrow (dG'\Psi/dn \leq 0). \)

**THEOREM 3':** If \( n = N_1 \) or \( N_2 \), then \( \partial \partial z \) [tax liability] = 0.

**PROOF:**
(i) The individual’s budget constraint may be written

\[
\text{tax liability} = T(z) + \sum_i Q'(x_i) + \sum_j \tau_j m_j(\cdot)
\]
where \( q = p + \tau. \)

(ii) Suppose that we increase labor income \( z \) by a small amount for some \( n \)-person on the optimal path. The other controls are held constant. Part of the income will be taxed and the rest will be spent. From (i) we have

\[
\frac{\partial}{\partial z} \text{tax liability} = 1 - \sum_j \frac{p_j}{u_j} \frac{\partial m_j(\cdot)}{\partial z}
\]

(iii) From Condition I we have

\[
0 = \frac{\partial H}{\partial z} = \mu \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial n} \right) + \frac{\partial}{\partial z} \left( \sum_j \frac{p_j}{u_j} \frac{\partial m_j(\cdot)}{\partial z} \right) f \text{ for all } i
\]

(iv) Combining (ii) and (iii) gives

\[
\frac{\partial \text{tax liability}}{\partial z} = -\mu \left\{ \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial n} \right) \right\}
\]

(v) By Condition VI we have \( \mu(N_1) = \mu(N_2) = 0 \); this fact and (iv) establishes the theorem.

**THEOREM 4':** \( \partial \text{[tax liability]} / \partial z \geq 0 \) if
(i) \( \partial u_j / \partial c_j \geq 0 \text{ all } j \)
(ii) \( d(G'\Psi)/dv \leq 0 \)

**PROOF:**
(i) From the proof of Theorem 3’ we have \( \partial \text{[tax liability]} / \partial z = -\mu/\lambda f \partial (\partial u/\partial n) / \partial z \).

(ii) \( \mu \leq 0 \) by the preceding Lemma; \( \lambda > 0 \); hence we only need to prove \( \{ \cdot \} \geq 0 \).

(iii) Repeat step (iii) of the proof to Theorem 4 to get \( \partial [\partial u/\partial n] / \partial z \geq 0 \) by using the assumption on the sign of the cross partials.

**THEOREM 5:** \( (u_{ik} = 0) \Rightarrow (\partial \text{tax liability}/\partial z = 0) \)

**PROOF:**
(i) Following steps (i)-(iii) of the proof to Theorem 3’, we obtain

\[
-\mu \frac{\partial}{\partial x_k} \left( \frac{\partial u}{\partial n} \right) = \lambda \left( -\frac{p_k}{\lambda f} \sum_j m_{jk} \right) f \text{ all } k
\]

Combining the above

\[
\frac{\partial \text{[tax liability]}}{\partial x_i} = -\frac{p_k}{\lambda f} \sum_j m_{jk}
\]

(ii) \( \partial u / \partial n \) = \( u_i z/n^2 \) by expanding the derivative

\[
\Rightarrow \frac{\partial [\cdot]}{\partial x_k} = 0 \text{ assuming } u_{ik} = 0 \text{ all } k
\]

(iii) Theorem 5 follows from (i) and (ii)

*Lemma invoked in Theorem 6a:*

\[ \partial [\partial u/\partial n] \partial q = -\Psi m_{nk} \]

**PROOF:** (see Mirrlees, 1976)
(i) Define the indirect utility function \( u^* \):
\[
\begin{align*}
\text{max} & \quad [u(-z/n,x,c):\Sigma qjcj &\leq m] \\
\text{by (12) and condition IV} & \quad \frac{\partial}{\partial q_k} \left( \frac{\partial u^*}{\partial q_k} \right) \\
\text{Rewrite the preceding expression in simpler notation:} & \quad \frac{z}{n^2} (u^*_{k1} + u^*_{m1} m_k)
\end{align*}
\]

(ii) Roy's Rule: \( u^*_{qk} = -\Psi m_k \)

(iv) Combine (ii) and (iii).

\[
\begin{align*}
\frac{\partial}{\partial q} \left[ \frac{\partial u}{\partial n} \right] & = \frac{z}{n^2} \left[ \frac{\partial}{\partial (-z/n)} (-\Psi c_{kj}) \right] \\
& + u^*_{im} m_k \text{ for } v \text{ constant} \\
& = \frac{z}{n^2} \left[ -\Psi \frac{\partial (m_k(\cdot))}{\partial (-z/n)} \right] \\
& - m_k u^*_{im} + u^*_{im} m_k \\
& = -\Psi m_k \\
\text{using } m_q = c_q \text{ and } \phi \Psi /\phi (-z/n) = u^*_{im} = u^*_{im}
\end{align*}
\]

THEOREM 6a: \((1 - w_k)m_k = \left. \frac{\partial R}{\partial \tau_k} \right|_{xz} \)

PROOF:

(i) By VIII we have
\[
0 = \frac{\partial J}{\partial q_k} = \int_{N_1}^{N_2} \left\{ \mu \frac{\partial u}{\partial q_k} \right\} dn - \lambda \sum_j p_j m_j f \text{ all } k
\]

(ii) By the preceding Lemma, we have:
\[
\frac{\partial}{\partial q_k} \left( \frac{\partial u}{\partial q_k} \right) = -\Psi m_k
\]

(iii) \( \sum_j p_j m_{kj} = \sum_j (q_j - \tau_j)m_{kj} \) since \( p_j = q_j - \tau_j \) all \( j \)

but \( \sum_j q_j m_{kj} = 0 \) (See John Hicks, p. 311.)

\[
\therefore \Sigma_j p_j m_{kj} = - \sum_j \tau_j m_{kj}
\]

(iv) The change in tax revenue is
\[
\frac{\partial R}{\partial q_k} \bigg|_{xz} = \int_{N_1}^{N_2} \frac{\partial m_k}{\partial q_k} + m_k f \text{ dn}
\]
where [\( \cdot \)] = tax liability by definition

\[
= \int_{N_1}^{N_2} (m_k + \sum_j \tau_j m_{kj}) f \text{ dn}
\]

(v) Combine (i), (ii), (iii), and (iv) to get
\[
0 = - \int_{N_1}^{N_2} \frac{\mu \Psi}{\lambda} m_k f \text{ dn}
\]

The theorem follows from \( v \) and our definitions of \( m_k \) and \( w_k \).

THEOREM 6c:

PROOF:

Rearrange terms in Theorem 6a and use the fact that
\[
\frac{\partial m(\cdot)}{\partial q_k} = c_k \quad \rightarrow \quad \int_{N_1}^{N_2} m_k(\cdot) f \text{ dn}
\]

THEOREM 6c:

PROOF:

(i) Combine steps (i)–(iii) in the proof of Theorem 6a to obtain
\[
0 = \int_{N_1}^{N_2} \left( -\mu \psi m_k + \lambda \sum_j \tau_j m_{kj} f \right) \text{ dn}
\]

(ii) By symmetry of the Slutsky matrix we have \( m_{jk} = m_{kj} \).
(iii) By definition
\[ w_k = - \int_{N_1}^{N_2} \frac{\mu \Psi}{\lambda} m_{kn} dn / \bar{m}_k \]
\[ \bar{m}_k = \bar{c}_k \]
\[ \therefore w_k \bar{c}_k = - \int_{N_1}^{N_2} \frac{\mu \Psi}{\lambda} m_{kn} dn \]

(iv) Combine (i), (ii), and (iii) to get Theorem 6c.

THEOREM 6d: \( \theta_k = \frac{w_k}{\eta} \)

PROOF:
(i) Rewrite Theorem 6c on the assumption that cross-price effects are nil:
\[ \int_{N_1}^{N_2} \tau_k m_{kk} f dn / \bar{c}_k = w_k \]

(ii) Define \( \theta_k = \tau_k / p_k \), the expenditure tax on good k
\[ \bar{\eta} = - \int_{N_1}^{N_2} \left( \frac{q_k}{\bar{c}_k} \frac{\partial c_k}{\partial q_k} \right) f dn \]
a measure of average elasticity of compensated demand.

(iii) Assume in the pretax situation \( p_k = q_k \); use this fact and (i) and (ii) to obtain Theorem 6d.

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